

# ISOTOPIES IN 3-MANIFOLDS<sup>(1)</sup>

BY

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**Introduction.** In the years immediately following the elegant proofs of Alexander [1] and Kneser [7] on the realization of homeomorphisms through deformations, relatively little work on isotopic deformations appeared. The reason lies, no doubt, in the inherent difficulty of constructing homeomorphisms. One needs to know a good deal about the structure of the space, preferably linear structure, before constructing or extending homeomorphisms. Also the space needs to have rather strong homogeneity conditions, like those of a manifold, before isotopy considerations become fruitful. These restrictions are severe, especially when, in several respects, the questions that arise become interesting only in three or more dimensions. With recent increases in the knowledge of 3-manifolds such questions have become approachable, and therefore it is, perhaps, not unnatural to find a number of papers devoted to the theme of isotopy appearing at this time (e.g. see [11; 12; 4] and [5]).

The main result of this paper is the following:

**THEOREM.** *Suppose  $M$  is a 3-manifold with boundary having triangulation  $\Sigma$  and  $\rho$  is the natural metric for  $\Sigma$ . Then for any  $\epsilon > 0$  there is a  $\delta > 0$  so that if  $f$  and  $g$  are homeomorphisms of  $M$  onto itself and  $\rho(f(x), g(x)) < \delta$  for all  $x$  in  $M$ , there is an  $\epsilon$ -isotopy of  $M$  taking  $g$  onto  $f$ <sup>(2)</sup>.*

This generalizes Sanderson's result [12] where  $M$  is  $E^3$ . For  $M$  compact, G. M. Fisher [4] has also obtained this result in an investigation following different lines than this one follows. For  $M$  compact, M.-E. Hamstrom [5] has announced the generalization that  $\mathcal{H}(M)$  under the compact-open topology is locally  $n$ -connected for each  $n$ .

Several consequences of the theorem are also given, including a characterization of those homeomorphisms of a compact 3-manifold with boundary which are isotopic to the identity. Another corollary is the theorem above with  $M$  a 2-manifold, and one could easily obtain the result for 2-manifolds with boundary by imitating all the ideas in the proof of the theorem one dimension lower.

**DEFINITIONS.** By  $\mathcal{H}(M)$  we designate the set of all homeomorphisms of the space  $M$  onto itself.  $\mathcal{H}(M, S)$  is the subset of  $\mathcal{H}(M)$  leaving the set  $S \subseteq M$

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<sup>(2)</sup> This result was announced in [6] and a brief outline of the proof was given there.

pointwise fixed. An *isotopy* of  $M$  is a continuous map  $H: M \times I \rightarrow M$  so that if we define  $H_t$  in the following manner:  $H_t(x) = H(x, t)$ , then for all  $t \in I$ ,  $H_t \in \mathcal{K}(M)$ . Usually we shall use  $H_t$  to designate the isotopy. If  $f$  and  $g$  are in  $\mathcal{K}(M)$   $f$  is *isotopic* to  $g$  if there is an isotopy  $H_t$  with  $H_0 = f$ ,  $H_1 = g$ . We say  $H_t$  *takes  $g$  onto  $f$*  if  $H_0 = 1$ , the identity, and  $H_1 g = f$ . If  $S$  is a subset of  $M$  and  $h$  is a homeomorphism of  $S$  into  $M$  then  $h(S)$  *can be taken isotopically onto  $S$*  means there is an isotopy  $H_t$  of  $M$  with  $H_0 = 1$ , and  $H_1(h(x)) = x$ , for all  $x \in S$ .

If  $M$  has a metric  $\rho$  an isotopy  $H_t$  of  $M$  is an  $\epsilon$ -isotopy if for each  $x$  in  $M$  and  $t_1, t_2$  in  $I$ ,  $\rho(H_{t_1}(x), H_{t_2}(x)) < \epsilon$ . If  $M$  is a geometric complex (locally-finite) a particularly useful metric is the *natural metric* [2] based on barycentric coordinates which assigns a length of  $2^{1/2}$  to each 1-simplex and makes each 2-simplex an equilateral triangle and each 3-simplex a regular tetrahedron. The function  $\rho^*: \mathcal{K}(M) \times \mathcal{K}(M) \rightarrow [0, \infty]$  defined as  $\rho^*(f, g) = \sup_{x \in M} \rho(f(x), g(x))$  will be useful.  $\rho^*$  behaves somewhat like a metric and is right invariant, i.e.  $\rho^*(fh, gh) = \rho^*(f, g)$ , hence if  $H_t$  is an  $\epsilon$ -isotopy then for any  $f \in \mathcal{K}(M)$  it follows that  $H_t f$  is also an  $\epsilon$ -isotopy.

A *3-manifold with boundary*  $M$  is a separable metric space with the property that each point has a neighborhood whose closure is a 3-cell. The *interior* of  $M$  ( $\text{Int } M$ ) consists of those points having a neighborhood homeomorphic with  $E^3$  and the boundary of  $M$  ( $\text{Bd } M$ ) is  $M - \text{Int } M$ . By a *triangulation*  $\Sigma$  of  $M$  we mean that  $M$  can be regarded as a locally-finite geometric complex and then  $\Sigma$  is the collection of all simplexes. A *polyhedron* in a geometric complex  $K$  is a set which is the sum of a collection of closed simplexes of some subdivision of  $K$ .

A set  $S$  in a geometric complex  $K$  is *tame* if there is a homeomorphism in  $\mathcal{K}(K)$  taking  $S$  onto a polyhedron. A homeomorphism  $h$  of a complex  $K$  into a complex  $L$  is *piecewise linear* (pwl) if for some subdivisions  $K'$  and  $L'$  of  $K$  and  $L$ , respectively,  $h$  carries every simplex in  $K'$  linearly onto a simplex in  $L'$ . An isotopy  $H_t$  is a *piecewise linear isotopy* if for each  $t$  in  $I$ ,  $H_t$  is pwl.

**Preliminary lemmas.** Before starting the proof of the theorem we state a number of lemmas and give proofs for those not in the literature.

**LEMMA 0.** *If  $C$  is any 3-cell with metric  $\rho$  and  $\epsilon > 0$ , there is a  $\delta > 0$  so that if  $h \in \mathcal{K}(C, \text{Bd } C)$  and  $\rho^*(1, h) < \delta$ , then there is an  $\epsilon$ -isotopy  $H_t$  taking  $h$  onto 1, and for each  $t$ ,  $H_t \in \mathcal{K}(C, \text{Bd } C)$ .*

**Proof.** Because  $C$  is homeomorphic to the unit ball  $B$  in  $E^3$  under a uniformly continuous transformation it suffices to prove the Lemma for  $B$ . Now see the remark after Theorem 1 in [6]. This is substantially the Alexander isotopy theorem [1].

**LEMMA 1 (SANDERSON).** *If  $L$  is a compact polyhedral 2-manifold in a triangulated 3-manifold  $M$  and  $\epsilon > 0$ , there is a  $\delta > 0$  so that if  $f$  is a pwl homeomorphism of  $L$  into  $M$  moving no point more than  $\delta$ , then there is a pwl  $\epsilon$ -*

*isotopy of  $M$  taking  $f(L)$  onto  $L$  and moving no point outside an  $\epsilon$ -neighborhood of  $L$ .*

For a proof of Lemma 1 cf. [11, Theorem 1].

**LEMMA 2 (BING).** *If  $U$  and  $U'$  are open subsets of triangulated 3-manifolds with boundary,  $f$  is a homeomorphism of  $U$  onto  $U'$  and  $a(x)$  is a positive continuous function on  $U$ , then there is a locally piecewise linear homeomorphism  $g$  of  $U$  onto  $U'$  and  $\rho(f(x), g(x)) < a(x)$  for  $x$  in  $U$ .*

A homeomorphism  $g$  of a complex  $C$  into a complex  $C$  is *locally pwl* at a point  $x$  if some neighborhood of  $x$  lies in a subcomplex on which  $g$  is pwl. We shall use the fact that a locally pwl homeomorphism maps a polyhedron into a polyhedron. For the proof of this and Lemma 2 see [2].

**LEMMA 3 (MOISE).** *Let  $K$  be a polyhedral 3-manifold with boundary and  $L$  a compact polyhedral 2-manifold in  $\text{Bd } K$ . Then  $L$  has a Cartesian product neighborhood (i.e. there is a homeomorphism  $g$  of  $L \times I$  into  $K$  so that  $g(L \times I) \cap \text{Bd } K = L$  and  $g(p \times 0) = p$  for each  $p$  in  $L$ ).*

This is only a slight modification of Lemma 7 in [8] as the compactness of  $K$  is not essential in the proof given there, and the containing space can be taken as the manifold one gets by attaching two copies of  $K$  by the identity map on their boundaries.

From Lemmas 1–3 a generalization in one direction of Lemma 1 is obtained.

**LEMMA 4.** *If  $L$  is a tame compact 2-manifold in  $E^3$  and  $\epsilon > 0$  there is a  $\delta > 0$  so that if  $h$  is a homeomorphism of  $L$  into  $E^3$  moving no point more than  $\delta$  and if  $h(L)$  is tame then there is an  $\epsilon$ -isotopy of  $E^3$  taking  $h(L)$  onto  $L$ .*

**Proof.** By the tameness of  $L$  there is no loss in generality in assuming  $L$  is polyhedral to begin with. First we shall show that if  $h$  is any homeomorphism of  $L$  so that  $h(L)$  is tame then  $h$  can be extended to a homeomorphism on an open set containing  $L$ . We start out by getting a Cartesian product neighborhood of  $L$  using Lemma 3 twice, since the closure of each component of  $E^3 - L$  is a 3-manifold with boundary. This allows us to coordinatize a neighborhood  $U$  of  $L$  in  $E^3$  by  $(x, t)$ , where  $x$  runs over  $L$  and  $t$  over the open interval  $(-1, 1)$ , and  $(x, 0)$  is the point  $x$  in  $E^3$ . Since  $h(L)$  is tame we can get such a neighborhood  $U'$  for it coordinatized by  $(x', t)$  where  $x'$  runs over  $h(L)$  and  $t$  over  $(-1, 1)$ . Extend  $h$  to  $h^*$  on  $U$  to  $U'$  as  $h^*(x, t) = (h(x), t)$ .

Next for any  $\eta > 0$ , we can get a positive function  $b(z) < \eta$  defined on  $h(U)$  such that  $b(z) \rightarrow 0$  as  $z$  approaches  $E^3 - h(U)$ . Let  $a(y) = b(h^*(y))$  and by Lemma 2 find  $g$  of  $U$  onto  $h(U)$  where  $g$  is locally pwl and  $\rho(h^*(x), g(x)) < a(x)$ . Then  $g(L)$  is polyhedral and  $gh^{-1}$  defines a homeomorphism of  $h(U)$  onto  $h(U)$  which moves no point more than  $\eta$  and takes  $h(L)$  onto a poly-

hedral 2-manifold. Also, since  $\rho(z, gh^{-1}(z)) \rightarrow 0$  as  $z \rightarrow E^3 - h(U)$  we can extend  $gh^{-1}$  to  $f$  in  $\mathcal{H}(E^3)$  by making  $f$  the identity outside  $h(U)$ .

The proof now runs as follows: if  $h$  is a homeomorphism that moves no point of  $L$  very far and  $h(L)$  is tame we find  $f$  in  $\mathcal{H}(E^3)$  very close to the identity so that  $fh|L$  is pwl. Next get a small isotopy  $F_t$  in  $E^3$  taking 1 onto  $f$  see [6, Theorem 1]. Now apply Lemma 1 to get a small isotopy  $G_t$  taking  $F_1h(L)$ , which is polyhedral, onto  $L$ . If  $H_t$  is the result of doing  $F_t$  followed by  $G_t$  then  $H_t$  takes  $h(L)$  onto  $L$ . The remainder of the argument is merely epsilontics and is left out.

REMARK. Instead of assuming tameness of  $h(L)$  in Lemma 4 and then extending  $h$  to an open set  $U$  containing  $L$ , we could have supposed  $h$  was defined on some open set  $U$  containing  $L$  to begin with. Then  $f$ , as defined above, took  $h(L)$  onto a polyhedron showing tameness. For this argument  $L$  need not be a polyhedral manifold but only a polyhedron.

A slightly stronger version of Lemma 4 is needed when  $L$  is a 2-sphere. Define, for  $t > 0$ ,  $B(t)$  to be the set of all  $(x_1, x_2, x_3)$  in  $E^3$  satisfying  $|x_i| \leq t, i = 1, 2, 3$ , in other words the cube of side  $2t$ . Define  $S(t)$  to be its boundary in  $E^3$ , a polyhedral 2-sphere.

LEMMA 5. *If  $\epsilon > 0$  there is a  $\delta > 0$  so that if  $h$  is a homeomorphism taking  $S(1 - \epsilon)$  into a tame sphere in  $B(1)$  and moving no point more than  $\delta$  then there is an  $\epsilon$ -isotopy  $H_t$  of  $E^3$  taking  $h(S(1 - \epsilon))$  onto  $S(1 - \epsilon)$  and fixed outside  $B(1)$ . It follows that  $H_1^{-1}|B(1)$  is an extension of  $h$  to all of  $B(1)$  which is the identity on  $S(1)$ .*

**Proof.** Since Lemma 1 will handle the case when  $h$  is pwl we only have to find a small isotopy  $F_t$  of  $E^3$  fixed outside  $B(1)$  so that  $F_0 = 1$  and  $F_1h$  is pwl. This is done as in Lemma 4 by finding  $f$ , a homeomorphism of  $E^3$  fixed outside  $B(1)$  which does not move points very far. Then in  $B(1)$  use Lemma 0 to take 1 onto  $f|B(1)$  with an isotopy  $F'_t$ . Then  $F_t$  is the isotopy  $F'_t$  on  $B(1)$  and 1 on  $E^3 - B(1)$ .

LEMMA 6. *Let  $D(r)$  denote the disk in  $E^2$  of radius  $r$ , center at 0. If  $\epsilon > 0$  there is a  $\delta$  so that if  $h$  is a homeomorphism of  $D(1 - \epsilon) \times I$  into  $D(1) \times I$  which is the identity on  $D(1 - \epsilon) \times ([0, \epsilon] + [1 - \epsilon, 1])$ , and if  $h$  moves no point more than  $\delta$ , then  $h|D(1 - 2\epsilon) \times I$  can be extended to  $h^*$  in  $\mathcal{H}(D(1) \times I)$ ,  $\rho^*(h^*, 1) < \epsilon$  and  $h^*$  will be the identity on the boundary of  $D(1) \times I$ .*

**Proof.** Let  $C(r)$  denote the boundary of  $D(r)$  in  $E^2$  and let  $D(1) \times I$  be in  $E^3$ . We first want to show that for  $h$  defined on and within  $\epsilon/3$  of the identity on  $D(1 - \epsilon) \times I$  that the torus  $L = h(C(1 - 2\epsilon) \times I) + C(1) \times I + (D(1) - D(1 - 2\epsilon)) \times (0 + 1)$  is tame. But if the open set  $U$  is defined by taking the  $\epsilon/3$ -neighborhood of  $L' = C(1 - 2\epsilon) \times I + C(1) \times I + (D(1) - D(1 - 2\epsilon)) \times (0 + 1)$  and  $h'$  is a homeomorphism of  $U$  into  $E^3$  which is  $h$  wherever  $h$  is defined, and 1 elsewhere, then since  $h'(L') = L$ , it follows from the remark after Lemma 4 that  $L$  is tame.

We will be done when we further restrict  $h$  so that we can extend  $h|L'$  to a homeomorphism defined on  $L' + \text{Int } L'$ , which moves no point more than  $\epsilon$ . Lemma 4 assures us of a  $\delta$  so that if  $h$  is within  $\delta$  of 1, then  $h'$  is within  $\delta$  of 1, and  $h'(L')$  can be taken by an  $\epsilon$ -isotopy  $H_t$  onto  $L'$ . Then  $H_t^{-1}|L' + \text{Int } L'$  is the desired extension.

**LEMMA 7.** *For any  $\epsilon > 0$  there is a  $\delta$  so that if  $h$  is a homeomorphism of  $K' = D(1 - \epsilon) \times [0, 1 - \epsilon]$  into  $K = D(1) \times I$  taking  $F' = D(1 - \epsilon) \times 0$  into  $F = D(1) \times 0$  and moving no point more than  $\delta$ , then there exists a  $g$  in  $\mathcal{H}(K)$  with  $\rho^*(g, 1) < \epsilon$ ,  $g|K'' = h$  and  $g|Bd K - F = 1$ , where  $K'' = D(1 - 2\epsilon) \times [0, 1 - 2\epsilon]$ .*

**Proof.** Let  $S$  denote the 2-sphere  $Bd(K - K'')$ , and  $F'' = D(1 - 2\epsilon) \times 0$ . As usual we think of  $K$  as imbedded in  $E^3$ . Note that if  $h$  takes  $K'$  into  $K$ ,  $F'$  into  $F$  and moves no point as much as  $\epsilon$ , then we almost get a homeomorphism  $h'$  of  $S$  into  $K$  when we define  $h'$  on  $Bd K'' - F''$  to be  $h$  and on  $Bd K - F$  to be 1.  $h'$  will be a homeomorphism on  $S$  as soon as we define it on the annulus  $A = F - F''$  into  $A' = F - h(F'')$ . In defining it there we also want  $h'$  close to the identity, so we need a 2-dimensional analog of Lemma 5. Such a result has been announced in [10] and also follows from conformal mapping theory (e.g. [3, Lemma 2]). We shall assume this result and obtain a homeomorphism  $h'$  of  $S$  as close to 1 as we like by restricting  $h$  close to 1 on  $K'$ . To see that  $h'(S)$  is tame, note that  $h' \text{ Cl}(Bd K'' - F'')$  is a tame disk, since  $h$  can be extended a little to an open set containing  $\text{Cl}(Bd K'' - F'') = D_1$  and  $h|D_1 = h'|D_1$  (the remark after Lemma 4 applying to a disk also). It is clear that  $h' \text{ Cl}(Bd K - F'')$  is tame. Now use the result of Moise [9] that the sum of two tame disks intersecting along their boundaries is a tame 2-sphere.

Now the hypothesis of Lemma 4 is set up, and for any  $\epsilon$  we can find a  $\delta'$  so that if  $h'$  on  $S$  is within  $\delta'$  of 1, we can extend  $h'$  to the interior of  $S$  getting, say,  $h^*$  on  $S + \text{Int } S$  which is within  $\epsilon$  of 1. Now go back and choose  $\delta$  so small that if  $h$  is within  $\delta$  of 1 on  $K'$  then  $h'$  as defined above will be within  $\delta'$  of 1 on  $S$ . Define  $g = h^*$  on  $S + \text{Int } S$  and  $g = h$  on  $K''$ , and the proof of Lemma 7 is complete.

**LEMMA 8.** *Let  $C = I^3$ , a 3-cube, and  $\epsilon > 0$ . There is a  $\delta$  so that if  $h$  is a homeomorphism of  $C' = [\epsilon, 1 - \epsilon] \times I \times I$  into  $C$  which is the identity on the three faces  $[\epsilon, 1 - \epsilon] \times I \times 0$ ,  $[\epsilon, 1 - \epsilon] \times I \times 1$ , and  $[\epsilon, 1 - \epsilon] \times 1 \times I$  and takes  $F' = [\epsilon, 1 - \epsilon] \times 0 \times I$  into  $F = I \times 0 \times I$ , then there is a  $g$  in  $\mathcal{H}(C)$  such that  $\rho^*(g, 1) < \epsilon$ ,  $g|Bd C - F = 1$  and  $g|C'' = h|C''$  where  $C'' = [2\epsilon, 1 - 2\epsilon] \times I \times I$ .*

**Proof.** The argument resembles that of Lemma 7. If  $h$  is within  $\epsilon$  of the identity and satisfies the other conditions of Lemma 8, we can define an  $h'$  on  $S' = Bd([0, 2\epsilon] \times I \times I)$  into  $C$  as soon as we extend the homeomorphism  $f$  on  $A_1 = 0 \times 0 \times I$ ,  $A_2 = 2\epsilon \times 0 \times I$ ,  $A_3 = [0, 2\epsilon] \times 0 \times 0$  and  $A_4 = [0, 2\epsilon] \times 0 \times 1$  (defined as  $f = 1$  on  $A_1 + A_3 + A_4$ ,  $f = h$  on  $A_2$ ) to  $f^*$  of the disk  $D$  in  $F$  bounded

by  $A_1 + A_2 + A_3 + A_4$  into the disk in  $F$  bounded by  $A_1 + h(A_2) + A_3 + A_4$ . Again we want the extension  $f^*$  close to 1 if  $f$  is. Again recourse is made to conformal mapping and a proof of just this fact is given in [3].

To define  $h'$  on  $S'$  we let it be the identity on  $0 \times I \times I + [0, 2\epsilon] \times I \times 0 + [0, 2\epsilon] \times I \times 1 + [0, 2\epsilon] \times 1 \times I$ ,  $h'$  is  $f^*$  on  $D$  and  $h' = h$  on  $2\epsilon \times I \times I$ . That  $h'(S')$  is tame is seen from the fact that  $h(2\epsilon \times I \times I)$  is a tame disk, as is the closure of the complement in  $h'(S')$ .

Now by restricting  $h$  close to 1 on  $C'$ ,  $h'$  on  $S'$  will be also near 1 and we can use Lemma 4 again to get an  $\epsilon$ -extension  $H'$  of  $h'$  to  $\text{Int } S'$ . Do the same for  $S'' = \text{Bd}([1 - 2\epsilon, 1] \times I \times I)$  getting a homeomorphism  $H''$  defined on  $S'' + \text{Int } S''$  which agrees with  $h$  on  $(1 - 2\epsilon) \times I \times I$ , moves no point more than  $\epsilon$  and is the identity on four faces of  $S''$ . Finally, if we define  $g$  as  $H'$  on  $S' + \text{Int } S'$ , as  $h$  on  $C'$  and as  $H''$  on  $S'' + \text{Int } S''$  we satisfy the conclusion of Lemma 8.

We are now ready to take up the proof of the Theorem.

**Proof of the Theorem.** It will first be assumed that  $M$  is a 3-manifold, i.e.,  $\text{Bd } M = \emptyset$ . Since the function  $\rho^*$ , which behaves like a metric on  $\mathcal{H}(M)$ , is right invariant, we need consider only the identity and homeomorphisms nearby under  $\rho^*$ . For if we can prove the Theorem for  $f = 1$  then  $\rho^*(f, g) < \delta$  implies that  $\rho^*(1, gf^{-1}) < \delta$  and if  $H_t$  is an  $\epsilon$ -isotopy taking  $gf^{-1}$  onto 1 then  $H_t f$  is an  $\epsilon$ -isotopy taking  $g$  onto  $f$ . Thus the uniformity in the Theorem is immediate.

The proof will consist of four stages each of which will further simplify (and reduce the class of) the homeomorphisms that we need to consider.

**STAGE 1.** Order the tetrahedra occurring in the triangulation  $\Sigma$  of  $M$  as  $T_1, T_2, T_3, \dots$ . Under the natural metric each  $T_i$  has edges of length  $2^{1/2}$ . For small  $\epsilon > 0$  let  $T_i^\epsilon$  be the unique tetrahedron contained in  $T_i$  having edge length  $=(1 - \epsilon)2^{1/2}$ , having the same barycenter as  $T_i$  and in the same relative position (i.e. the faces of  $T_i^\epsilon$  are parallel to those of  $T_i$  using the linear structure in  $T_i$ ). Let  $\epsilon_1 > 0$  be fixed. Since there is a homeomorphism of  $T_1$  onto the cube  $B(1)$  taking  $T_1^\epsilon$  onto  $B(1 - \epsilon_1)$ , it follows from Lemma 5 that there is a  $\delta_1$  so that if  $h$  is in  $\mathcal{H}(M)$  and  $\rho^*(h, 1) < \delta_1$ , then for each integer  $i$  there is a homeomorphism  $h_i^*$  of  $T_i$  onto itself such that  $h_i^*|T_i^\epsilon = h$ ,  $h_i^*$  moves no point as much as  $\epsilon_1$  and  $h_i^*| \text{Bd } T_i = 1$ . Let  $h^*$  in  $\mathcal{H}(M)$  be defined as  $h^*|T_i = h_i^*$ . Now applying Lemma 0 to each  $T_i$  we obtain an  $\epsilon_1$ -isotopy  $H_i^t$  of  $T_i$  fixed on  $\text{Bd } T_i$  taking  $h_i^*$  onto 1. Let  $H_t$  be the  $\epsilon_1$ -isotopy of  $M$  taking  $h^*$  onto 1, defined as  $H_t|T_i = H_i^t$ . The net result of applying  $H_t$  to  $h$  is that with an  $\epsilon_1$ -isotopy we deform  $h$  to  $H_1 h = h^{*-1} h$ , and  $h^{*-1} h$  moves no point more than  $\delta_1 + \epsilon_1$  and is the identity on  $T_i^\epsilon$  for each  $i$ .

Put another way, by choosing  $\epsilon_1$  to be small and  $\delta_1$  still smaller, and by restricting  $h$  in  $\mathcal{H}(M)$  to lie within  $\delta_1$  of 1, we can with an  $\epsilon_1$ -isotopy take  $h$  onto a homeomorphism  $g$ , and

(i)  $g$  is the identity on  $M$  except in an  $\epsilon_1$ -neighborhood of the 2-skeleton of  $\Sigma$ .

In Stage 2 we shall restrict ourselves to those homeomorphisms satisfying (i) where  $\epsilon_1$  is as small as we choose.

STAGE 2. Order the 2-simplexes occurring in  $\Sigma$  as  $F_1, F_2, \dots$ . Each  $F_i$  is a face of exactly two tetrahedra  $T_{i1}, T_{i2}$ . For small  $\epsilon > 0$  let  $F_i^*$  be the unique concentric triangle in  $F_i$  having length  $(1 - \epsilon) \cdot 2^{1/2}$  and identified with  $F_i^* \times 0$  in  $N_i^*$ , which is a triangular prism  $F_i^* \times [-d(\epsilon), +d(\epsilon)]$  symmetric with respect to  $F_i$ , one half of  $N_i$  lying in  $T_{i1}$ , and the other half lying in  $T_{i2}$ . For each  $i$  we obtain  $N_i^*$ , forming a collection of isometric sets. The height  $2d(\epsilon)$  of each prism  $N_i$  is dependent only on  $\epsilon$ , and is chosen small enough so that  $N_i^*$  misses  $N_j^*$  if  $i \neq j$ .

Choose  $\epsilon_2 > 0$ . Since  $N_i^*$  is homeomorphic with  $D(1) \times I$ , we can use Lemma 6 to find a  $\delta_2$  so that if  $\rho^*(g, 1) < \delta_2$  and  $g$  is the identity on the top and bottom of  $N_i^*$  (which it will be if in (i) we take  $\epsilon_1$  small compared to  $\epsilon_2$ ) then there is a homeomorphism  $g_i^*$  of  $N_i^*$  which is  $g$  on  $F_i^{3\epsilon_2} \times [-d(\epsilon_2), d(\epsilon_2)]$ , the identity on  $\text{Bd } N_i^*$  and moves no point more than  $\epsilon_2$ . For each  $i$  find  $g_i^*$  and an isotopy  $G_i^*$  of  $N_i^*$  by Lemma 0 taking  $g_i^*$  onto the identity moving no point farther than say  $\epsilon'_2$ . Define  $g^*$  in  $\mathcal{K}(M)$  to be  $g_i^*$  on  $N_i^*$  and 1 everywhere else. Define  $G_i$ , an  $\epsilon'_2$ -isotopy of  $M$ , to be  $G_i^*$  on  $N_i^*$  and 1 elsewhere. Then  $G_i$  takes  $g^*$  onto 1, hence  $G_i$  takes  $g$  onto  $G_i g = g^{*-1}g$  and  $g^{*-1}g$  moves no point more than  $\delta_2 + \epsilon'_2$  and is the identity on  $F_i^{3\epsilon_2} \times [-d(\epsilon_2), d(\epsilon_2)]$ . In fact for any  $\epsilon_2 > 0$ , if we choose  $\epsilon_1$  small at Stage 1, then in Stage 2 we reduce  $g$  with an  $\epsilon_2$ -isotopy to a homeomorphism  $f$  of  $M$  and

(ii)  $f$  is the identity on  $M$  except in an  $\epsilon_2$ -neighborhood of the 1-skeleton of  $\Sigma$ .

In Stage 3 we restrict ourselves further to those homeomorphisms satisfying (ii), where  $\epsilon_2$  is as small as we choose.

STAGE 3. Order the 1-simplexes of  $\Sigma$ ,  $E_1, E_2, \dots$ . For small  $\epsilon > 0$  let  $E_i^*$  be the segment of length  $(1 - \epsilon) \cdot 2^{1/2}$  in  $E_i$  having the same midpoint. At this point we would like to get a canonical neighborhood of  $E_i^{2\epsilon}$  for each  $i$ , thereby obtaining a collection of isometric sets as we did in Stage 2. This is impossible since we cannot give the precise linear structure of the star of an edge (i.e. the sum of all tetrahedra containing an edge), the linear structure, of course, depending on the number of tetrahedra in the star.

However we are far enough along in the deformation that we no longer need isometric sets. Let  $x$  be any point of  $E_i^*$  and  $r(\epsilon)$  small compared to  $\epsilon$  (anything less than  $\epsilon/2$  will do). For each tetrahedron  $T_j$  in  $\Sigma$  containing  $E_i$  there is a unique isosceles triangle  $P_j^*(x)$  which is perpendicular to  $E_i$  at  $x$ , contained in  $T_j$  and its equal sides lie in  $\text{Bd } T_j$  and have length  $r(\epsilon)$ . The sum of all  $P_j^*(x)$ , where  $T_j$  contains  $E_i$ , is a polyhedral disk  $D_i^*(x)$  and the sum of all  $D_i^*(x)$ , for  $x$  in  $E_i^*$ , is a polyhedral cylinder  $C_i^*$  containing  $E_i^{2\epsilon}$  on its interior. We can think of  $C_i^*$  as  $D_i^* \times E_i^{2\epsilon}$ . Now note that the diameter of each of the two components of  $C_i^* - D_i^* \times E_i^{2\epsilon}$  is small, in fact smaller than  $2\epsilon$ . Denote the closures of these components by  $Q_i^*$  and  $Q_i^*$ .

Fix  $\epsilon_3 > 0$ . Now if  $f$  is a homeomorphism in  $\mathcal{K}(M)$  satisfying (ii) for  $\epsilon_2$  small

compared to  $\epsilon_3$ , then  $f$  will be the identity on  $\text{Bd } D_i^3 \times E_i^3$ . We want to replace  $f$  on a  $C_i^3$  by a homeomorphism  $f_i^*$  agreeing with  $f$  on  $D_i^3 \times E_i^{2\epsilon_3}$  but which is 1 on  $\text{Bd } C_i^3$ . Because of the small diameters of  $Q_1^3$  and  $Q_2^3$  it will follow that  $f_i^*$  moves no point far. To get such an  $f_i^*$  we merely have to redefine  $f$  on  $Q_1^3$  and  $Q_2^3$  by extending the map on, say,  $\text{Bd } Q_1^3$  defined as 1 on  $\text{Bd } Q_1^3 - D_i^3 \times E_i^{2\epsilon_3}$  and as  $f$  on  $Q_1^3 \cap D_i^3 \times E_i^{2\epsilon_3}$  to all of  $Q_1^3$ . Do the same for  $Q_2^3$ .

We can find an isotopy  $F_i^t$  on  $C_i^3$  taking  $f_i^*$  onto the identity, fixed on  $\text{Bd } C_i^3$  and moving no point farther than  $\rho^*(f_i^*, 1)$ . The method is essentially that of Lemma 0 where care has to be taken to insure the last condition, which can be done as follows.

In the cylinder  $C_i^3 = D_i^3 \times E_i^3$  consider the smaller but similar cylinder  $C_i^t(t) = D_i^t \times E_i^{(\epsilon^{-1})^{t+1}}$  for  $0 < t \leq 1$ , i.e. a cylinder whose "radius" and "length" is just  $t$  times that of  $C_i^3$ . There is an obvious homeomorphism  $h_t$  of  $C_i^3$  onto  $C_i^t(t)$  which contracts distances, (i.e.  $\rho(x, y) \geq \rho(h_t(x), h_t(y))$  for  $x$  and  $y$  in  $C_i^3$ ) using line segments emanating from the mid-point of  $E_i$ . Then  $F_i^t$  is defined as  $h_t f_i^* h_t^{-1}$  wherever  $h_t^{-1}$  is defined and 1 elsewhere. The last condition on  $F_i^t$  above follows as a consequence to the contracting nature of  $h_t$ .

Proceed in the same way for each edge  $E_i$ . Define  $f^*$  to be  $f_i^*$  on  $C_i^3$  and 1 elsewhere, define  $F_t$  to be  $F_i^t$  on each  $C_i^3$  and 1 elsewhere. Then  $F_t$  takes  $f^*$  onto 1 hence  $f$  onto  $F_t f = f^{*-1} f = e$  and if  $\epsilon_2$  is sufficiently small,  $F_t$  is an  $\epsilon_3$ -isotopy of  $M$  and

(iii)  $e$  is the identity on  $M$  except in an  $\epsilon_3$ -neighborhood of the vertices in  $\Sigma$ .

Hence in Stage 4 we restrict ourselves still further to those homeomorphisms satisfying (iii) where  $\epsilon_3$  can be as small as we choose.

**STAGE 4.** Let  $V_1, V_2, \dots$  be the vertices in  $\Sigma$ . Take the barycentric subdivision  $\Sigma'$  of  $\Sigma$  and let  $B_i$  be the sum of all tetrahedra in  $\Sigma'$  having  $V_i$  as a vertex.  $B_i$  is a 3-cell. Clearly the  $B_i$  are disjoint. Since  $B_i$  is starlike with respect to  $V_i$ , (i.e. the intersection of  $B_i$  with every radial line emanating from  $V_i$  is a line segment, using linear structure of tetrahedra in  $\Sigma$ ), if a homeomorphism on  $B_i$  is the identity on  $\text{Bd } B_i$  and is within  $\epsilon$  of 1, then it can be taken by an  $\epsilon$ -isotopy back to 1. Here the Alexander deformation of Lemma 0 takes place along the radial lines, much as in Stage 3.

Finally if  $e$  is a homeomorphism of  $M$  satisfying (iii) for  $\epsilon_3$  small compared to the diameters of the  $B_i$  then  $e|_{\text{Bd } B_i} = 1$ , each  $i$ , and  $e_i = e|_{B_i}$  can be taken onto the identity in  $\mathcal{H}(B_i)$  by a  $2\epsilon_3$ -isotopy  $E_i^t$ . Define  $E_t$  to be  $E_i^t$  on  $B_i$  and 1 elsewhere and  $E_t$  takes  $e$  onto 1. The proof of the Theorem for 3-manifolds is now complete.

If  $M$  is a 3-manifold with boundary, the proof requires the following modifications. Let  $L = \text{Bd } M$ . First we require of  $\Sigma$  that every tetrahedron in  $\Sigma$  intersect  $L$  in a 2-simplex, edge, vertex or not at all. If  $\Sigma$  is not such a triangulation already we can obtain one by taking cones over boundaries of tetrahedra from interior points of tetrahedra. If we can prove the theorem for



the natural metric of this new triangulation we can prove it for  $\Sigma$  so we might as well suppose  $\Sigma$  has the property to begin with.

In Stage 1 we modify the interior of a tetrahedron  $T_i$  having a face  $F$  in  $L$  differently. Instead of employing Lemma 5, we use Lemma 7 to get  $h_i$  in  $\mathcal{K}(T_i)$  which is 1 on  $\text{Bd } T_i - F$  and is the restriction of  $h$  on a tetrahedron in  $\text{Int } T_i + F$ , one of whose faces lies in  $F$  and whose edges are  $(1 - \epsilon_1) \cdot 2^{1/2}$  in length. The isotopy  $H_i^1$  is obtained by first deforming  $h_i^*|F$  onto 1 by Alexander's result in 2-dimensions, and extending to  $T_i$  along radial lines emanating from the barycenter of  $T_i$  leaving points in  $\text{Bd } T_i - F$  fixed, and then deforming  $h_i^*$  the rest of the way onto 1. We piece together  $H_i$  as before and  $H_i$  takes  $h$  onto  $g$  which, besides satisfying (i), satisfies an additional condition

(i')  $g$  is the identity on  $L$  except in an  $\epsilon_1$ -neighborhood of the 1-skeleton of the triangulation of  $L$  induced by  $\Sigma$ .

In Stage 2 we modify  $g$  near those 2-simplexes in  $\Sigma$  which do not lie in  $L$  just as before. The 2-simplexes in  $L$  are left alone. The resulting isotopy, which is 1 except near the altered faces, takes  $g$  onto  $f$  and  $f$  satisfies (ii).

In Stage 3 we modify  $f$  slightly differently near an edge  $E_i$  of  $\Sigma$  in  $L$ . We get a canonical Cartesian product neighborhood  $C_i^3 = D_i^3 \times E_i^{2\epsilon_3}$  of  $E_i^{2\epsilon_3}$ , much as before, except of course this time  $E_i^{2\epsilon_3}$  lies in a disk  $F$  on the  $\text{Bd } C_i^3$ , where  $F$  is composed of two subdisks, one from each of the 2-simplexes in  $L$  having  $E_i$  as edge. Now Lemma 8 is employed to get  $f_i^*$  on  $C_i^3$  where  $f_i^*$  is 1 on  $\text{Bd } C_i^3 - F$ , and  $f$  on  $D_i^3 \times E_i^{2\epsilon_3}$  and within  $\epsilon_3$  of the identity. The small isotopy  $F_i^1$  of  $C_i^3$  that takes  $f_i^*$  onto 1 is defined in two steps. First we define a small isotopy on  $F$  taking  $f_i^*|F$  onto 1 and this isotopy is extended to one on  $C_i^3$  leaving points on  $\text{Bd } C_i^3 - F$  fixed. Then we deform  $f_i^*$  the rest of the way onto 1 by a second use of Alexander's isotopy. Piece together  $F_i$  as before and  $F_i$  takes  $f$  onto  $e$  and  $e$  satisfies (iii) as before.

Stage 4 is the same except for a slight modification near a vertex  $V_i$  lying in  $L$ . Then at the conclusion of Stage 3, no matter how small we make  $\epsilon_3$ ,  $e$  will not be 1 on  $\text{Bd } B_i$ . However we can force  $e$  to be 1 there except on an arbitrarily small disk. The small isotopy  $E_i^1$  is then defined in two steps, the first taking  $e$  into a homeomorphism  $e'$  which is 1 on the  $\text{Bd } B_i$ , the second taking  $e'$  onto 1. This finishes the proof of the Theorem.

**Corollaries to the theorem.** Denote those homeomorphisms in  $\mathcal{K}(M)$  which are isotopic to the identity by  $\mathcal{I}(M)$ . If  $M$  is a compact 3-manifold with boundary we have shown that  $\mathcal{K}(M)$  is uniformly locally-0-connected under the topology in  $\mathcal{K}(M)$  induced by  $\rho^*$  (which is a metric in the compact case and induces the compact-open topology) and hence the subgroup  $\mathcal{I}(M)$  contains 1 as an interior point in  $\mathcal{K}(M)$ , hence  $\mathcal{I}(M)$  is an open subgroup of  $\mathcal{K}(M)$ . Also, because of the correspondence between isotopies and arcs in the  $C-O$  topology,  $\mathcal{I}(M)$  is the arc component of 1 in  $\mathcal{K}(M)$  and hence it is connected.

**COROLLARY 1.** *If  $M$  is a compact 3-manifold with boundary then  $h \in \mathcal{G}(M)$  iff  $h = h_k \cdot \dots \cdot h_2 h_1$ , where  $h_i$  is the identity outside a polyhedral 3-cell,  $i = 1, 2, \dots, k$ .*

**Proof.** Because of Alexander's result, any  $h_i$  which is the identity outside a 3-cell  $C$  can be deformed onto 1 by an isotopy of  $C$ , which can then be extended to an isotopy of  $M$ . This proves sufficiency, since  $\mathcal{G}(M)$  is a subgroup.

To show necessity we will show that some neighborhood  $N$  of 1 has the property that if  $h \in N$  then  $h = h_k \cdot \dots \cdot h_2 h_1$  where  $h_i$  is the identity outside a 3-cell  $C_i$ . Let  $\epsilon = 1$  and find a  $\delta$  by the construction used in the Theorem, so that if  $\rho^*(h, 1) < \delta$  then there is an  $\epsilon$ -isotopy taking  $h$  onto 1, and the  $\epsilon$ -isotopy is that obtained by first doing  $H_i$ , followed by  $G_i$ , followed by  $F_i$ , followed finally by  $E_i$ . Hence,  $1 = E_1 F_1 G_1 H_1 h$ , where  $H_1 = h^{*-1}$ ,  $G_1 = g^{*-1}$ ,  $F_1 = f^{*-1}$  and  $E_1 = e^{-1}$ , so that  $h = h^* g^* f^* e$ . Further, each of  $e, f^*, g^*$  and  $h^*$  are products of finitely many homeomorphisms fixed outside a polyhedral 3-cell. E.g.  $e$  is the product of the  $e'_i$ , where  $e'_i$  is defined to be  $e_i^*$  on  $B_i$  and 1 elsewhere;  $h^*$  is the product of the  $h'_i$ , where  $h'_i$  is defined to be  $h_i^*$  on  $T_i$  and 1 elsewhere etc. This proves Corollary 1 since an open set containing the identity generates the whole group if the group is connected.

**COROLLARY 2.** *If the hypothesis is the same as in the theorem and  $\epsilon > 0$  there is a  $\delta$  so that if  $\rho^*(h, 1) < \delta$  and  $h$  is the identity outside a closed set  $K$  then there is an  $\epsilon$ -isotopy taking  $h$  onto 1 which does not move any point outside an  $\epsilon$ -neighborhood of  $K$ .*

**Proof.** Take a barycentric subdivision  $\Sigma^n$  of  $\Sigma$  whose mesh is less than  $\epsilon/4$ . Let  $\rho_1$  be the metric assigning a length of  $2^{1/2}/2^n$  to a 1-simplex in  $\Sigma^n$  and making simplexes regular in  $\Sigma^n$ . The identity in  $\mathcal{H}(M)$  then is uniformly continuous from  $\rho$  to  $\rho_1$  and vice versa. The proof of Corollary 2 proceeds as in the Theorem, using tetrahedra in  $\Sigma^n$  and metric  $\rho_1$  and building canonical neighborhoods as before, however in Stage 1, if  $h|_{T_i} = 1$ , where  $T_i$  is a tetrahedron in  $\Sigma^n$ , no modification of  $h$  on  $T_i$  is made in getting  $g$ . Similarly in Stage 2, if  $g$  is already the identity on the Cartesian product cube constructed on part of the interior of the 2-simplex  $F_i$ , then  $g$  is not modified in getting  $f$  etc. In brief, all our modifications are done economically.

As a result, if  $\rho_1(x, K) \geq \epsilon$  no chain of 4 tetrahedra in  $\Sigma^n$  exist whose sum is connected, contains  $x$  and intersects  $K$ . But the homeomorphism at the start of Stage  $i$  will not be modified during Stage  $i$  on a tetrahedron  $T$  unless it is not the identity on some tetrahedron intersecting  $T$  ( $i = 2, 3, 4$ ) hence the  $\epsilon$ -isotopy  $H_i$  thus constructed leaves  $x$  fixed throughout.

**COROLLARY 3.** *If  $L$  is a tame compact 2-manifold in any 3-manifold  $M$  and  $\epsilon > 0$ , there is a  $\delta > 0$  so that if  $h$  is a homeomorphism of  $L$  into  $M$  moving no point more than  $\delta$  and if  $h(L)$  is tame, then there is an  $\epsilon$ -isotopy of  $M$  taking  $h(L)$  onto  $L$  and moving no point outside an  $\epsilon$ -neighborhood of  $L$ .*

**Proof.** There is no loss in generality in supposing  $L$  polyhedral and a subset of the 2-skeleton of a triangulation  $\Sigma$  of  $M$ . We shall construct a Cartesian product neighborhood of  $L$  as follows. First we define a new manifold with boundary  $M'$  by splitting  $M$  along  $L$ , i.e. each point of  $L$  becomes two points of  $M'$ . Then  $M'$  (which may be disconnected) has just two boundary components, say  $L_1$  and  $L_2$ . Lemma 3 allows us to get a Cartesian product neighborhood of each of  $L_1$  and  $L_2$  in  $M'$ , so that when we re sew  $L_1$  and  $L_2$  together to get  $M$  again we can obtain an imbedding of  $L \times [-1, 1]$  in  $M$  where the points  $(p, 0)$  and  $p$  correspond for each  $p \in L$ .

This means that we can extend any homeomorphism taking  $L$  into a tame set to an open set containing  $L$ . As in the proof of Lemma 4 we can take  $h(L)$  onto a polyhedron by  $g$  in  $\mathcal{H}(M)$  which is arbitrarily close to 1 and moves only those points near  $h(L)$ . By Corollary 2,  $g$  can be obtained by a small isotopy moving only points near  $h(L)$ . Lemma 1 allows us to deform  $gh(L)$  onto  $L$  with a small isotopy moving only points near  $L$  providing  $gh$  is close to 1. Putting the two isotopies together gives us the desired deformation. The rest is epsilonotics.

**COROLLARY 4.** *If  $M$  is a 3-manifold and  $\epsilon > 0$ , there is a  $\delta > 0$  so that if  $h$  is a homeomorphism of the 2-skeleton  $K$  of  $\Sigma$  into  $M$ , moving no point more than  $\delta$  and such that  $h(K)$  is tame, then there is an  $\epsilon$ -isotopy of  $M$  taking  $h(K)$  onto  $K$ .*

**Proof.** Using Corollary 3, extend  $h|_{\text{Bd } T}$  to a homeomorphism close to the identity on  $T$ , a tetrahedron in  $\Sigma$ . Do this for each  $T$  getting an extension of  $h$  to all of  $M$ . Now use the Theorem.

**COROLLARY 5.** *The theorem holds if  $M$  is a 2-manifold.*

**Proof.**  $M \times [0, 1)$  is a 3-manifold with boundary which can be triangulated so that the natural metric of  $M \times [0, 1)$  induces the natural metric on  $M = M \times 0$ . Any homeomorphism of  $M \times [0, 1)$  onto itself takes  $M$  onto  $M$ .

That even Corollary 5 is false for an arbitrary metric can be seen by taking a 2-sphere, removing a point and adding a sequence of handles converging to the missing point: This can be done in  $E^3$  and the metric  $\rho$  used will be that of  $E^3$ . Define homeomorphisms  $\{h_n\}$  where  $h_n$  consists of cutting around the  $n$ th handle, twisting a full turn in the  $1/n$ -neighborhood of the cut and reattaching. Then  $\{h_n\} \rightarrow 1$  under  $\rho^*$  but for no  $n$  is  $h_n$  isotopic to 1.

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